Homework Problems I
PHYS 425: Electromagnetism I

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1. Griffiths Problem 1.4

The three vertices of the triangle are 1= $\hat{x}$, 2=2 $\hat{y}$, and 3=3 $\hat{z}$. The vector pointing from vertex 1 to vertex 2 is $\boldsymbol{v}_1$ = 2 $\hat{y}$ - $\hat{x}$. The vector pointing from vertex 2 to vertex 3 is $\boldsymbol{v}_2$ = 3 $\hat{z}$ -2 $\hat{y}$. The vector pointing from vertex 3 to vertex 1 is $\boldsymbol{v}_3$ = $\hat{x}$ - 3 $\hat{z}$. Using the component form of cross product, it is verified that

$$\boldsymbol{v}_1 \times \boldsymbol{v}_2 = \boldsymbol{v}_2 \times \boldsymbol{v}_3 = \boldsymbol{v}_3 \times \boldsymbol{v}_1 = 6\hat{x} + 3\hat{y} + 2\hat{z},$$

is the normal direction. The unit normal is

$$\frac{6\hat{x} + 3\hat{y} + 2\hat{z}}{\sqrt{36+9+4}} = \frac{6\hat{x} + 3\hat{y} + 2\hat{z}}{7}.$$

2. Griffiths Problem 1.13

(a)

$$\begin{aligned}
&\nabla\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right] = \\
&2(x-x')\hat{x} + 2(y-y')\hat{y} + 2(z-z')\hat{z} \\
&= 2\boldsymbol{\mathscr{r}}
\end{aligned}$$

(b)

$$\begin{aligned}
&\nabla\left(\frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}\right) \\
&= -\frac{1}{2}\frac{1}{\left[\sqrt{(x-x')^2 + (y-y'')^2 + (z-z')^2}\right]^3}\left[2(x-x')\hat{x} + 2(y-y')\hat{y} + 2(z-z')\hat{z}\right] \\
&= -\frac{1}{\boldsymbol{\mathscr{r}}^2}\frac{\left[(x-x')\hat{x} + (y-y')\hat{y} + (z-z')\hat{z}\right]}{\sqrt{(x-x')^2 + (y-y'')^2 + (z-z')^2}} = -\frac{\hat{\boldsymbol{\mathscr{r}}}}{\boldsymbol{\mathscr{r}}^2}
\end{aligned}$$

(c) For *n* positive or negative

$$\begin{aligned}
&\nabla \boldsymbol{\mathscr{r}}^n = n\boldsymbol{\mathscr{r}}^{n-1}\nabla\boldsymbol{\mathscr{r}} \\
&= n\boldsymbol{\mathscr{r}}^{n-1}\frac{1}{2}\frac{\left[2(x-x')\hat{x} + 2(y-y')\hat{y} + 2(z-z')\hat{z}\right]}{\sqrt{(x-x')^2 + (y-y'')^2 + (z-z')^2}} \\
&n\boldsymbol{\mathscr{r}}^{n-1}\hat{\boldsymbol{\mathscr{r}}}
\end{aligned}$$

Note the cases (a) and (b) work, and that in general negative powers will have a negative result.

3. Griffiths Problem 1.27

$$\nabla\cdot\left[\nabla\times\boldsymbol{v}\right]=\nabla\cdot\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ v_x & v_y & v_z\end{vmatrix}$$

$$=\frac{\partial}{\partial x}\left(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}\right)+\frac{\partial}{\partial y}\left(\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)$$

$$=0$$

by the equality of the mixed partial derivatives.

$$\boldsymbol{v}_a=x^2\hat{x}+3xz^2\hat{y}-2xz\hat{z}$$

$$\nabla\times\boldsymbol{v}_a=\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x^2 & 3xz^2 & -2xz\end{vmatrix}=\left(0-6xz\right)\hat{x}+\left(0-\left(-2z\right)\right)\hat{y}+\left(3z^2-0\right)\hat{z}$$

$$\nabla\cdot\nabla\times\boldsymbol{v}_a=-6z+0+6z=0$$

Griffiths Problem 1.29 (for free, I did the wrong problem!)

$$\boldsymbol{v}=x^2\hat{x}+2yz\hat{y}+y^2\hat{z}$$

(a)

$$\int\boldsymbol{v}\cdot d\boldsymbol{l}=\int_0^1 x^2dx+\int_0^1 2(1)(0)\,dy+\int_0^1(1)^2\,dz$$

$$\frac{1}{3}+0+1=\frac{4}{3}$$

(b)

$$\int\boldsymbol{v}\cdot d\boldsymbol{l}=\int_0^1\left[(t)^2(1)+2tt+(t)^2\right]dt\qquad\vec{x}(t)=(t,t,t)$$

$$4\frac{t^3}{3}\bigg|_0^1=4\frac{1}{3}=\frac{4}{3}$$

(c)

$$\int \boldsymbol{v}\cdot d\boldsymbol{l} = \int_0^1 x^2 dx + \int_0^1 2(1)(0)\,dy + \int_0^1 (1)^2\, dz$$

$$\frac{1}{3}+0+1=\frac{4}{3}$$

(d) $\oint \boldsymbol{v}\cdot d\boldsymbol{l} = \frac{4}{3}-\frac{4}{3}=0$

One should suspect the vector function is a gradient. Indeed

$$\boldsymbol{v} = \nabla f\left(x,y,z\right) = \nabla\left(\frac{x^3}{3}+y^2 z\right)$$

$$f\left(1,1,1\right)=\frac{4}{3} \qquad f\left(0,0,0\right)=0$$

4. Griffiths Problem 1.30

Defining the normal for the surface integral to be $\hat{z}$

$$\boldsymbol{v} = 2xz\hat{x} + \left(x+2\right)\hat{y} + y\left(z^2-3\right)\hat{z}$$

$$\int_F \boldsymbol{v}\cdot\hat{n}da = \int_0^2\int_0^2 \left(0-3\right)ydxdy = -6\int_0^2 ydy = -12$$

As the total flux from the other five faces in Ex. 1.7 is 20, it is clear that the surface integral does NOT depend only on the boundary line. Two different surfaces with the same boundary line integrate to different results.

The total flux through the closed surface of the cube is (recognizing the outward normal for the bottom face is $-\hat{z}$)

$$20-\left(-12\right)=32$$

Looking ahead, this means that $\boldsymbol{v}$ cannot be the curl of another vector function because if it was, the first corollary to Stokes Theorem would imply independence on the surface used to evaluate the integral. Indeed

$$\nabla\cdot\boldsymbol{v} = 2z+2zy \neq 0,$$

and the volume integral of the divergence gives

$$\int_C \nabla\cdot\boldsymbol{v}dV = \int_0^2\int_0^2\int_0^2 \left(2z+2yz\right)dxdydz = 4\int_0^2\int_0^2\left(z+yz\right)dydz = 4\int_0^2\left(2z+2z\right)dz = 16\frac{4}{2} = 32.$$

By the Divergence theorem, this integral must sum to the total flux through the cube surfaces, which it does.

5. Griffiths Problem 1.35

First note that the orientation of the line integral in Ex. 1.11 is such that the normal is the inward normal for the back cube face. By the corollary and Stokes Theorem applied as in Ex. 1.11, it should agree with the sum of the surface integrals obtained with the outward normal vectors of the five other cube faces. The five surface integrals are

$$\nabla \times \mathbf{v} = \left(4z^2 - 2x\right)\hat{x} + 4yz^2\hat{z}$$

$$\int_{(i)} \nabla \times \mathbf{v} \cdot \hat{n} da = \int_0^1\int_0^1 \left(4z^2 - 2(1)\right) dydz = \frac{4}{3} - 2$$

$$\int_{(ii)} \nabla \times \mathbf{v} \cdot \hat{n} da = -\int_0^1\int_0^1 2(0)\, dxdy = 0$$

$$\int_{(iii)} \nabla \times \mathbf{v} \cdot \hat{n} da = \int_0^1\int_0^1 (0)\, dxdz = 0$$

$$\int_{(iv)} \nabla \times \mathbf{v} \cdot \hat{n} da = \int_0^1\int_0^1 (0)\, dxdz = 0$$

$$\int_{(v)} \nabla \times \mathbf{v} \cdot \hat{n} da = \int_0^1\int_0^1 2(1)\, dxdy = 2$$

The sum is 4/3, the same as the flux into the remaining face as calculated in Ex. 1.11.

6. Griffiths Problem 1.39

(a)

$$\nabla \cdot \left(r^2\hat{r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r} r^4 = 4r$$

$$\int_S \nabla \cdot \left(r^2\hat{r}\right) dV = \int_0^R\int_{-1}^1\int_0^{2\pi} 4rr^2 dr d\cos\theta d\varphi = 4\pi R^4$$

$$\int_{\partial S} r^2\hat{r} \cdot \hat{r} da = R^2 4\pi R^2$$

(b)

$$\nabla\cdot\left(\frac{\hat{r}}{r^2}\right)=\frac{1}{r^2}\frac{\partial}{\partial r}1=0$$

$$\int_S \nabla\cdot\left(\frac{\hat{r}}{r^2}\right)dV=\int_0^R\int_{-1}^{1}\int_0^{2\pi}0drd\cos\theta d\varphi=0$$

$$\int_{\partial S}\frac{\hat{r}\cdot\hat{r}}{r^2}da=4\pi R^2/R^2=4\pi$$

Strictly speaking, as in the discussion of the Dirac delta, the divergence for the inverse square law is not defined at the origin, as the field blows up there!

7. Griffiths Problem 1.45

(a)

$$\int_{-2}^{2}(2x+3)\delta(3x)dx=\int_{-2}^{2}(2x+3)\frac{\delta(x)}{3}dx=\frac{2(0)}{3}+\frac{3}{3}=1$$

(b)

$$\int_0^2\left(x^3+3x+2\right)\delta(1-x)dx=(1)^3+3(1)+2=6$$

(c)

$$\int_{-1}^{1}9x^2\delta(3x+1)dx=\int_{-1}^{1}9x^2\,\frac{\delta(x+1/3)}{3}dx=\frac{9(-1/3)^2}{3}=\frac{1}{3}$$

(d)

$$\int_{-\infty}^{a}\delta(x-b)dx=\begin{cases}0 & a<b\\ 1 & a>b\end{cases}$$

8. Griffiths Problem 1.49

$$\int_V e^{-r}\nabla\cdot\left(\frac{\hat{r}}{r^2}\right)dV=4\pi\int_V e^{-r}\delta^3(\mathbf{r})dV=4\pi e^0=4\pi$$

because the origin is included in the integration volume (a sphere centered at the origin).

Using 1.59

$$\int_S e^{-r}\nabla\cdot\left(\frac{\hat{r}}{r^2}\right)dV=-\int_S\nabla\left(e^{-r}\right)\cdot\left(\frac{\hat{r}}{r^2}\right)dV+\int_{\partial S}e^{-r}\,\frac{\hat{r}}{r^2}\cdot\hat{n}da$$

$$=\int_S\left(\frac{e^{-r}}{r^2}\right)dV+e^{-R}\,\frac{4\pi R^2}{R^2}=4\pi\int_0^R e^{-r}dr+4\pi e^{-R}=-4\pi e^{-R}+4\pi+4\pi e^{-R}=4\pi$$

9. Griffiths Problem 1.50

$$\nabla \cdot \boldsymbol{F}_1 = 0+0+0 = 0$$

$$\nabla \times \boldsymbol{F}_1 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & x^2 \end{vmatrix} = \hat{x}(0-0) + \hat{y}(0-2x) + \hat{z}(0-0) = -2x\hat{y}$$

$$\nabla \cdot \boldsymbol{F}_2 = 1+1+1 = 3$$

$$\nabla \times \boldsymbol{F}_2 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \hat{x}(0-0) + \hat{y}(0-0) + \hat{z}(0-0) = 0$$

$\boldsymbol{F}_1$ has a vector potential

$$\boldsymbol{V} = \frac{x^2}{4}\hat{z}$$

$$\boldsymbol{v} = \boldsymbol{V} \times \boldsymbol{r} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & \frac{x^2}{4} \\ x & y & z \end{vmatrix} = -\frac{x^2 y}{4}\hat{x} + \frac{x^3}{4}\hat{y}$$

$\boldsymbol{F}_2$ has a scalar potential

$$\phi(\boldsymbol{r}) = \int_0^{\boldsymbol{r}} \boldsymbol{F}_2 \cdot d\boldsymbol{l} = \int_0^1 txxdt + \int_0^1 tyydt + \int_0^1 tzzdt = \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2}$$

$\boldsymbol{F}_3$ has both a scalar potential and a vector potential

$$\nabla \cdot \boldsymbol{F}_3 = 0 + 0 + 0 = 0$$

$$\nabla \times \boldsymbol{F}_3 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix} = \hat{x}\left(x - x\right) + \hat{y}\left(y - y\right) + \hat{z}\left(z - z\right) = 0$$

$$\phi\left(\boldsymbol{r}\right) = \int_0^{\boldsymbol{r}} \boldsymbol{F}_3 \cdot d\boldsymbol{l} = \int_0^1 tytzxdt + \int_0^1 tztxydt + \int_0^1 txtyzdt = \frac{xyz}{3} + \frac{xyz}{3} + \frac{xyz}{3}$$

$$= xyz$$

$$\boldsymbol{V} = \frac{yz}{4}\hat{x} + \frac{zx}{4}\hat{y} + \frac{xy}{4}\hat{z}$$

$$\boldsymbol{v} = \boldsymbol{V} \times \boldsymbol{r} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{yz}{4} & \dfrac{zx}{4} & \dfrac{xy}{4} \\ x & y & z \end{vmatrix} = \frac{x\hat{x}}{4}\left(z^2 - y^2\right) + \frac{y\hat{y}}{4}\left(x^2 - z^2\right) + \frac{z\hat{z}}{4}\left(y^2 - x^2\right)$$

10. Griffiths Problem 1.57